

ON STRICTLY DEZA GRAPHS WITH PARAMETERS $(n, k, k - 1, a)$

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ABSTRACT. A nonempty k -regular graph Γ on n vertices is called a *Deza graph* if there exist constants b and a ($b \geq a$) such that any pair of distinct vertices of Γ has either b or a common neighbours. The quantities n , k , b , and a are called the parameters of Γ and are written as the quadruple (n, k, b, a) . If a Deza graph has diameter 2 and is not strongly regular, then it is called a *strictly Deza graph*. In the present paper, we investigate strictly Deza graphs whose parameters (n, k, b, a) satisfy the conditions $k = b + 1$ and $\frac{k(k-1) - a(n-1)}{b-a} > 1$.

Keywords: regular graphs, graphs with regularity conditions, Deza graphs, strictly Deza graphs

Dedicated to the memory of Michel Deza

1. INTRODUCTION

In the present paper, we consider finite undirected graphs without loops and multiple edges. A nonempty k -regular graph Γ on n vertices is called a *Deza graph* if there exist constants b and a such that any pair of distinct vertices of Γ has either b or a common neighbours. We assume further that $b \geq a$. The quantities n , k , b , and a are called the parameters of Γ and are written as the quadruple (n, k, b, a) .

The concept of a Deza graph was introduced in 1999 by M. Erickson, S. Fernando, W. Haemers, D. Hardy, and J. Hemmeter in the seminal paper [4] influenced by A. Deza and M. Deza [3]. Deza graphs generalize strongly regular graphs in the sense that the number of common neighbours of any pair of vertices in a Deza graph does not depend on adjacency.

A strongly regular graph has diameter 2, except for the trivial case of a disjoint union of complete graphs. As opposed to strongly regular graphs, Deza graphs can have diameter greater than 2. If a Deza graph has diameter 2 and is not strongly regular, then it is called a *strictly Deza graph*. So, we have a trihotomy for the class of Deza graphs: strongly regular graphs, strictly Deza graphs, and Deza graphs of diameter greater than 2.

In [4] a basic theory of strictly Deza graphs was developed and several ways to construct such graphs were introduced. Moreover, all strictly Deza graphs with number of vertices at most 13 were found. In 2011, the investigation of strictly Deza graphs was continued by S. Goryainov and L. Shalaginov in [5]. They found

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all strictly Deza graphs whose number of vertices is equal to 14, 15, or 16. In 2014, S. Goryainov and L. Shalaginov in [6] found all strictly Deza graphs that are Cayley graphs with number of vertices less than 60.

Problems arising in the theory of strictly Deza graphs sometimes are similar to problems in the theory of strongly regular graphs. However, results and methods in these theories differ. In our opinion, an analysis of these differences can enrich both theories.

For example, it is known that the connectivity of a connected strongly regular graph equals its valency [2]. In 2014, the connectivity of some strictly Deza graphs was investigated in [8]. In particular, an example of a strictly Deza graph whose connectivity and valency were not equal was found.

If Γ is a strongly regular graph, then its parameters are written as (n, k, λ, μ) , where λ is the number of common neighbours of every two adjacent vertices of Γ and μ is the number of common neighbours of every two nonadjacent and distinct vertices of Γ .

If a strongly regular graph Γ has parameters (n, k, λ, μ) such that $k = \mu$, then Γ is a complete multipartite graph with parts of size $n - k$ (see Section 1.3 in [1]). An analogue of this result for strictly Deza graphs with condition $k = b$ was also obtained in the above-mentioned paper [4].

The complement of a strongly regular graph with parameters (n, k, λ, μ) is also strongly regular with parameters $(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$. Therefore, if a strongly regular graph Γ has parameters (n, k, λ, μ) , where $k = \mu$, then the parameters of complement $\bar{\Gamma}$ satisfy the equality $\bar{k} = \bar{\lambda} + 1$. Hence, the structure of a strongly regular graph Γ with $k = \lambda + 1$ can be obtained from the corresponding result for a strongly regular graph with $k = \mu$ and vice versa.

It is important to note that there is the other situation in the case of strictly Deza graphs. Namely, let Γ be a strictly Deza graph. Its complement $\bar{\Gamma}$ is a Deza graph only if Γ is a coedge-regular graph with $b = a + 2$. Thus, there is no direct connections between a strictly Deza graph with parameters satisfying $k = b$ and a strictly Deza graph with parameters satisfying $k = b + 1$. The aim of this paper is to investigate strictly Deza graphs with parameters (n, k, b, a) satisfying the condition $k = b + 1$ which resemble to strongly regular graphs with $k = \lambda + 1$. The structure of such Deza graphs turned out to be much more complicated than the corresponding case of strongly regular graphs.

Let us introduce some definitions and notation.

Let Γ be a graph with the vertex set $V(\Gamma)$, and let $v \in V(\Gamma)$. The set of vertices adjacent to v is called the *neighbourhood of v* and is denoted by $N(v)$. The set $N(v) \cup \{v\}$ is called the *closed neighbourhood of v* and is denoted by $N[v]$. The set of vertices at distance 2 from a vertex v is called the *second neighbourhood of v* and is denoted by $N_2(v)$.

Let Δ_1 and Δ_2 be graphs. A graph Γ is called the *extension of Δ_1 by Δ_2* if the following conditions hold:

- (1) $V(\Gamma)$ is the set of pairs (v_1, v_2) such that $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$.
- (2) Vertices (v_1, v_2) and (u_1, u_2) are adjacent in Γ if and only if either v_1 and u_1 are adjacent in Δ_1 or $v_1 = u_1$ and v_2, u_2 are adjacent in Δ_2 .

Sometimes, such a graph Γ is called the *composition of Δ_1 and Δ_2* .

We say that Γ is the m -clique extension of Δ_1 if Δ_2 is the complete graph K_m on m vertices. We say that Γ is the m -coclique extension of Δ_1 if Δ_2 is the complement $\overline{K_m}$ of the complete graph on m vertices.

Now we introduce the following special notation for Deza graphs. Let Γ be a Deza graph with parameters (n, k, b, a) , and let v be a vertex of Γ . We consider the following subsets of $V(\Gamma)$:

$A(v)$ is the set of all vertices $u \in V(\Gamma)$ such that $|N(v) \cap N(u)| = a$.

$B(v)$ is the set of all vertices $u \in V(\Gamma)$ such that $|N(v) \cap N(u)| = b$.

$B[v] := B(v) \cup \{v\}$.

The number of vertices of $A(v)$ is denoted by $\alpha(v)$, and the number of vertices of $B(v)$ is denoted by $\beta(v)$. It is known that the numbers $\alpha(v)$ and $\beta(v)$ for a given strictly Deza graph Γ are constants independent from the choice of a vertex v (see [4, Proposition 1.1]). We denote these constants of Γ by $\alpha(\Gamma)$ and $\beta(\Gamma)$, respectively.

The main results of the present paper are the following two theorems.

Theorem 1. *Let Γ be a strictly Deza graph with parameters (n, k, b, a) and $\beta(\Gamma) > 1$. The parameters k and b of Γ satisfy the condition $k = b + 1$ if and only if Γ is isomorphic to the 2-clique extension of the complete multipartite graph with parts of size $\frac{n - k + 1}{2}$.*

Graphs from the conclusion of Theorem 1 are pointed out in [4] (see Example 2.4).

Theorem 2. *Let Γ be a strictly Deza graph with parameters $(n, k, k - 1, a)$ and $\beta(\Gamma) > 1$. Then $a = 2k - n$ and Γ is recognizable by its parameters.*

By Theorem 1 Γ is isomorphic to the 2-clique extension of the complete multipartite graph with parts of size $\frac{n - k + 1}{2}$. Hence, $a = n - 2(n - k + 1) + 2 = 2k - n$ and Γ has parameters $(n, k, k - 1, 2k - n)$.

In the forthcoming paper, we will consider strictly Deza graphs with parameters $(n, k, k - 1, a)$ and $\beta(\Gamma) = 1$.

2. PRELIMINARY RESULTS

Let Γ be a strictly Deza graph with parameters (n, k, b, a) . By definition, $V(\Gamma) = \{v\} \cup A(v) \cup B(v)$ for any vertex $v \in \Gamma$. Hence, it is easy to see that $n = 1 + \alpha(\Gamma) + \beta(\Gamma)$.

Lemma 1. *The following equality holds for $\beta(\Gamma)$:*

$$\beta := \beta(\Gamma) = \frac{k(k - 1) - a(n - 1)}{b - a}.$$

Proof. See Proposition 1.1 in [4]. □

Lemma 2. *A strictly Deza graph with parameters $(n, k, k - 1, k - 2)$ does not exist.*

Proof. Let Γ be a strictly Deza graph with parameters $(n, k, k - 1, k - 2)$. Since $b - a = 1$, then we have $\beta = k(k - 1) - (k - 2)(n - 1) > 0$ by Lemma 1.

Let $k = 2$. Then Γ is the cycle of length n , and since Γ has diameter 2 we have that $n \in \{4, 5\}$. But the cycles of length 4 and 5 are strongly regular graphs. Thus, no cycle can be a strictly Deza graph.

Let $k > 2$. Since $\beta > 0$, the following inequality holds:

$$n < \frac{k^2 - 2}{k - 2} = k + 2 + \frac{2}{k - 2}.$$

If $k = 3$, then $n < 8$. However, strictly Deza graphs having less than 8 vertices do not exist [4].

If $k > 3$, then $n \leq k + 2$. Since Γ is not a complete graph, we have $n = k + 2$. Hence, for every vertex v , there exists a unique nonadjacent vertex u such that all other vertices of Γ are adjacent both to v and u . However, $b = k$ in this case, and we have a contradiction to the condition $b = k - 1$. \square

3. PROOF OF THEOREM 1

Till the end of the proof, let Γ be a strictly Deza graph with parameters (n, k, b, a) , where $k = b + 1$ and $\beta = \frac{k(k-1) - a(n-1)}{b-a} > 1$.

At first, we consider all possibilities of mutual placement of sets $B(v)$ and $N(v)$ for an arbitrary vertex v of Γ .

Lemma 3. *Let $v \in V(\Gamma)$. Then, for the set $B(v)$, one of the following statements holds:*

- (1) $B(v) \cap N(v) = \emptyset$;
- (2) $B(v) \subset N(v)$;
- (3) $|B(v) \cap N(v)| = 1$.

Proof. Let us recall that if a vertex u belongs to $B(v)$, then $|N(u) \cap N(v)| = b = k - 1$. If $u \in N(v) \cap B(v)$, then $N(v) = \{u\} \cup (N(u) \cap N(v))$ and u is adjacent to no vertex outside the closed neighbourhood $N[v]$. If $w \in B(v) \setminus N[v]$, then there exist a unique vertex of $N(v)$ nonadjacent to w . Therefore, if neither (1) nor (2) hold, then $|N(v) \cap B(v)| = 1$. \square

We assume further that a vertex v of Γ is of *type* (A), (B), or (C) if $B(v)$ satisfies statement (1), (2), or (3) of Lemma 3, respectively.

Lemma 4. *For parameters of Γ , the following inequalities hold:*

- (1) $\alpha > 0$;
- (2) $b > a > 0$.

Proof. Since Γ is not a strongly regular graph, $\alpha \neq 0$ and $b \neq a$. Let $a = 0$. Since Γ has diameter 2, $A(v)$ is contained in $N(v)$ for each vertex v in Γ . Moreover, any two vertices of $A(v)$ are not adjacent.

Fix a vertex u of Γ . If a vertex u is of type (A), then $B(u) \cap N(u) = \emptyset$. Hence, $B(u) = \Gamma \setminus N[u]$ and $A(u) = N(u)$. Therefore, we have $\alpha = k$ and $\beta = n - k - 1$. Note that, from the equalities $a = 0$ and $\alpha = k$, we have $A(v) = N(v)$. Thus, $B(v) = \Gamma \setminus N[v]$ for each vertex v of Γ . However, it is impossible since Γ is not a strongly regular graph.

If u is a vertex of type (B) or (C) in Γ , then $|B(u) \cap N(u)| \neq \emptyset$. Since $A(u) \subset N(u)$, any vertex v of $N(u) \cap B(u)$ is adjacent to any vertex of $A(u)$. This contradicts the assumption $a = 0$. \square

By Lemma 4, it is obvious that, for any two vertices v and u of Γ , the intersection of their neighbourhoods $N(v) \cap N(u)$ cannot be empty.

In further Lemmas 5 – 11, our aim is to investigate properties of vertices of type (A).

Lemma 5. *Let x be a vertex of type (A) in Γ . If there exist two distinct vertices x_1 and x_2 in $B(x)$ such that $N(x) \cap N(x_1) = N(x) \cap N(x_2)$, then $N(x) \cap N(x_1) = N(x) \cap N(x_i)$ for each vertex x_i from $B(x)$.*

Proof. Since $b = k - 1$, we have $N(x) \cap N(x_1) = N(x) \cap N(x_2) = N(x_1) \cap N(x_2)$. Let x_i be a vertex from $B(x)$ and $x_i \notin \{x_1, x_2\}$. If $N(x) \cap N(x_1) \neq N(x) \cap N(x_i)$, then $|N(x) \cap N(x_1) \cap N(x_i)| = k - 2$. By Lemma 2, the parameter a is not equal to $k - 2$. Therefore, $|N(x_1) \cap N(x_i)| = |N(x_2) \cap N(x_i)| = k - 1$. However, there is a unique vertex in $N(x_i) \setminus N(x)$. Hence, we have $N(x_i) \setminus N(x) \subset N(x_1) \cap N(x_2)$. Thus, $|N(x_1) \cap N(x_2)| = k$. A contradiction with $|N(x_1) \cap N(x_2)| = k - 1$. \square

Lemma 6. *Let a vertex x be of type (A) in Γ , and let $B(x) = \{x_1, \dots, x_\beta\}$. Then the subgraph induced on $B[x]$ in Γ is a coclique of size $\beta + 1$.*

Proof. Let $x_i x_j$ be an edge of the subgraph induced on $B[x]$ in Γ . If $N(x) \cap N(x_i) \neq N(x) \cap N(x_j)$, then $|N(x_i) \cap N(x_j)| = k - 2 = a$, which is impossible by Lemma 2. Hence, $N(x) \cap N(x_i) = N(x) \cap N(x_j)$. Since $x_j \in N(x_i) \setminus N(x)$, the equality $N(x) \cap N(x_i) = N(x_i) \cap N(x_j)$ holds. Hence, there exists a vertex y of $N(x)$ such that $N(x) = \{y\} \cup (N(x_i) \cap N(x_j))$. Since $N(x) \cap B(x) = \emptyset$, we have $y \notin B(x)$ and $N(x) \setminus N[y] \neq \emptyset$.

Therefore, there exists a vertex v in $(N(x_i) \cap N(x_j)) \setminus N[y]$. Since x is a vertex of type (A), we have $v \notin B(x)$. In this case, $|N(v) \cap N(x)| = a$. Moreover, $N(v) \cap N(x)$ is contained in $N(x_i) \cap N(x_j)$. Hence, $N(v) \cap N(x_i)$ contains the set $N(v) \cap N(x)$ and the vertex x_j . Therefore, $N(v) \cap N(x_i)$ contains $a + 1 = b$ vertices. But this contradicts Lemma 2. \square

Lemma 7. *For each vertex x of type (A) and each vertex x_i in $B(x)$, the equality $B[x] = B[x_i]$ holds. Moreover, $N(x_i) \cap B(x_i) = \emptyset$. Thus, each vertex $x_i \in B(x)$ is of type (A).*

Proof. Let $x_i \in B(x)$. It is clear that $x \in B(x_i)$. For any vertex $x_j \in B(x)$ ($x_j \neq x_i$), the inequality $|N(x_i) \cap N(x_j)| \geq |N(x) \cap N(x_i) \cap N(x_j)| \geq k - 2$ holds. By Lemma 2 $|N(x_i) \cap N(x_j)| > k - 2$. Therefore, $|N(x_i) \cap N(x_j)| = k - 1$ and $x_j \in B(x_i)$. This implies the equality $B[x] = B[x_i]$.

Moreover, by Lemma 6, the subgraph induced on $B[x]$ in Γ is a coclique. Since $B[x] = B[x_i]$, we have $N(x_i) \cap B(x_i) = \emptyset$. Thus, each vertex $x_i \in B(x)$ is of type (A). \square

Remark. By the choice of a vertex x , for any vertex $x_i \in B(x)$, there is a unique vertex in $N(x)$ nonadjacent to x_i . At the same time, there is a unique vertex in $N_2(x)$ adjacent to x_i .

Lemma 8. *Suppose that x is a vertex of type (A), $y \in N(x)$, and y is nonadjacent to a vertex $x_i \in B(x)$. Let $z \in N_2(x)$ be adjacent to x_i . Then the following statements hold:*

- (1) $N(x) \cap N(y) = N(x) \cap N(z)$;
- (2) $B[v]$ is contained in $N(x) \cap N(y)$ for any vertex v from $N(x) \cap N(y)$.

Proof. (1) At first, since x is a vertex of type (A) and $y \notin B(x)$, we have $|N(y) \cap N(x)| = a$. Then $N(x) = \{y\} \cup (N(x) \cap N(x_i))$. Moreover, $N(x) \cap N(y) \subseteq N(x) \cap$

$N(x_i)$. By Lemma 4(2), we have $N(x) \cap N(y) \neq \emptyset$. If $v \in N(x) \cap N(y)$, then v is adjacent to the vertex y and $a - 1$ vertices in $N(x_i)$. But v must be adjacent to at least a vertices in $N(x_i)$. Hence, v is adjacent to z . Therefore, $N(x) \cap N(y)$ is contained in $N(z) \cap N(x)$. By Lemma 6, $z \notin B(x)$. Therefore, $|N(z) \cap N(x)| = a$. This implies the required equality $N(x) \cap N(y) = N(x) \cap N(z)$.

(2) Let $v \in N(x) \cap N(y)$ and $w \in B(v)$. By statement (1), we have $\{x, y, z, x_i\} \subseteq N(v)$. Since $w \in B(v)$, w is adjacent to at least three vertices in $\{x, y, z, x_i\}$. Now the required assertion follows from the equality $N(x) \cap N(y) = N(x) \cap N(z)$ proved in the previous statement and from the equality $N(x_i) \cap N(y) = N(x_i) \cap N(z)$ obtained from it by applying Lemma 7. \square

Lemma 9. *For any vertex x of type (A), one of the following statements holds:*

(1) *There is a unique vertex $y \in N(x)$ such that $\{y\} = N(x) \setminus N(x_i)$ for any $x_i \in B(x)$. Moreover, $|\bigcup_{i=1}^{\beta}(N(x_i) \setminus N(x))| = \beta$.*

(2) *There is a unique vertex $z \in N_2(x)$ such that $\{z\} = N(x_i) \setminus N(x)$ for any $x_i \in B(x)$. Moreover, $|\bigcup_{i=1}^{\beta}(N(x) \setminus N(x_i))| = \beta$.*

Proof. (1) Let x be a vertex of type (A). Then $B(x)$ is contained in $N_2(x)$. Since $k = b + 1$, we have $|N(x) \setminus N(x_i)| = 1$ and $|N(x_i) \setminus N(x)| = 1$. By the condition of Theorem 1, we have $\beta > 1$.

Let the number of vertices in $\bigcup_{i=1}^{\beta}(N(x) \setminus N(x_i))$ be less than β . Then there exist two distinct vertices x_i and x_j of $B(x)$ such that the equality $N(x) \cap N(x_i) = N(x) \cap N(x_j)$ holds. In this case, by Lemma 5, $N(x) \cap N(x_i) = N(x) \cap N(x_l)$ for any vertex x_l in $B(x)$. Therefore, there exists a vertex y such that $N(x) = \{y\} \cup N(x) \setminus N(x_i)$ for any x_i in $B(x)$. Moreover, for distinct vertices x_j and x_l of $B(x)$, any two differences $N(x_j) \setminus N(x)$ and $N(x_l) \setminus N(x)$ cannot coincide. Otherwise, $|N(x_j) \cap N(x_l)| = k$, but this contradicts the condition on the parameter b of Γ . Hence, the number of vertices in the set $\bigcup_{i=1}^{\beta}(N(x_i) \setminus N(x))$ is equal to the number of vertices in $B(x)$. Thus, the condition (1) for Γ holds.

(2) Let the differences $N(x) \setminus N(x_i)$ be pairwise distinct for $i \in \{1, \dots, \beta\}$. Therefore, $|N(x_i) \cap N(x_j) \cap N(x)| = k - 2$. Since $a \neq k - 2$ by Lemma 2, we have $|N(x_i) \cap N(x_j)| = b$ for any two distinct vertices x_i and x_j of $B(x)$. It follows that, for any pair $N(x_i)$ and $N(x_j)$, there exist a unique common vertex $z \in N(x_i) \cap N(x_j) \cap N_2(x)$. Thus, condition (2) holds for Γ . \square

Further, if the conclusion of statement (1) of Lemma 9 holds, let $N(x_i) \setminus N(x) = \{y_i\}$ and $\bigcup_{i=1}^{\beta}(N(x_i) \setminus N(x)) = \{y_1, \dots, y_{\beta}\}$. If the conclusion of statement (2) of Lemma 9 holds, let $N(x) \setminus N(x_i) = \{z_i\}$ and $\bigcup_{i=1}^{\beta}(N(x) \setminus N(x_i)) = \{z_1, \dots, z_{\beta}\}$.

Lemma 10. *If statement (1) of Lemma 9 holds, then $B(y) = \{y_1, \dots, y_{\beta}\}$ and y is of type (A).*

If statement (2) of Lemma 9 holds, then $B(z) = \{z_1, \dots, z_{\beta}\}$ and z is of type (A).

Proof. Let statement (1) of Lemma 9 hold. Let v be any vertex from $N(y) \setminus N[x]$. The vertex v cannot belong to $B(x)$ by the choice of y . Therefore, $|N(x) \cap N(v)| = a$. But $N(x) \cap N(v)$ contains y and, hence, only $a - 1$ vertices from $N(x) \cap N(x_i)$ for each $x_i \in B(x)$. However, v must be adjacent to at least a vertices in $N(x_i)$. Thus, v must be adjacent to a vertex in $N(x_i) \setminus N(x) = \{y_i\}$. Hence, any vertex from $N(y) \setminus N(x)$ is adjacent to y_i . By Lemma 8(1) any vertex y_i belongs to $B(y)$.

Since any vertex $y_i \notin B(x)$, we have $|N(y_i) \cap N(x)| = a$. Thus, $N(y_i)$ does not contain y and $N(y) \cap \{y_1, \dots, y_{\beta}\} = \emptyset$. Therefore, y is of type (A).

Let statement (2) of Lemma 9 hold. If we consider an arbitrary vertex v from $N(z_i) \setminus N[x]$, then similar arguments as in the previous case prove the lemma. \square

We proved that the vertices y and z in Lemma 10 are of type (A). Further, for convenience, if a vertex v has type (A), then we denote by v^* a unique vertex described in Lemma 9.

By Lemma 10, we have two possibilities for vertices of type (A) in Γ . Further, if a vertex satisfies statement (1) of Lemma 9, then we call it a *vertex of type (A1)*. If a vertex satisfies statement (2) of Lemma 9, then we call it a *vertex of type (A2)*.

Lemma 11. *For any vertex of type (A) of Γ , one of the following statements holds.*

(1) *If x is a vertex of type (A1), then $N(x) \setminus \{x^*\}$ contains $B[v]$ for every vertex $v \in N(x) \setminus \{x^*\}$.*

(2) *If x is a vertex of type (A2), then $N(x) \setminus B(x^*)$ contains $B[v]$ for every vertex $v \in N(x) \setminus B(x^*)$.*

Proof. (1) Let x be a vertex of type (A1), and let v be an arbitrary vertex from $N(x) \setminus \{x^*\}$. If $v \in N(x) \cap N(x^*)$, then $B[v]$ is contained in $N(x) \setminus \{x^*\}$ by Lemma 8(2). Let $v \in N(x) \setminus N[x^*]$. Thus, if w is an arbitrary vertex from $B(v)$, then $w \neq x^*$ by Lemma 9(1). Moreover, if w is not adjacent to x , then w is adjacent to all vertices of $B(x)$. Since $\beta > 1$, there exist two distinct vertices x_i and x_j in $B(x)$ such that w belongs to $N(x_i) \cap N(x_j)$. But $N(x_i) \cap N(x_j) \subset N(x)$ since x is a vertex of type (A1). A contradiction. Thus, w is adjacent to x and $B[v]$ is contained in $N(x) \setminus \{x^*\}$.

(2) Let x be a vertex of type (A2) and v be an arbitrary vertex from $N(x) \setminus B(x^*)$. Then v is adjacent to all vertices of $B[x]$. Let there exist a vertex w in $B(v)$ that does not belong to $N(x) \setminus B(x^*)$.

If w is not adjacent to x , then w is adjacent to a vertex x_i from $B(x)$. Hence, by Lemma 9(2), $w \in N(x_i) \setminus N(x) = \{x^*\}$. Since $w \in B(v)$, we have $v \in B(w) = B(x^*)$. A contradiction with the choice of v from $N(x) \setminus B(x^*)$. Thus, w is adjacent to x and $B[v]$ is contained in $N(x) \setminus N(x^*)$.

If $w \in N(x) \cap B(x^*)$, then here exist a unique vertex $x_i \in B(x)$ such that $w \notin N(x_i)$. However, $|N(x^*) \cap N(w)| = |N(v) \cap N(w)| = k - 1$. Thus, $|N(x^*) \cap N(v)| \geq k - 2$. It contradicts $v \notin B(x^*)$ and Lemma 2. Hence, $w \in N(x) \setminus B(x^*)$ and $B[v] \subseteq N(x) \setminus B(x^*)$. \square

In the next lemma, we study vertices of types (B) and (C) in Γ .

Lemma 12. (1) *Let v be a vertex of type (B) in Γ , and let u be any vertex from $B(v)$. Then $B[u] = B[v]$ and u is of type (B).*

(2) *Let v be a vertex of type (C) in Γ , and let u be any vertex from $B(v)$. Then $B[u] = B[v]$ and u is of type (C).*

Proof. (1) If v is a vertex of type (B) in Γ , then, for each $u \in B(v)$, we have $N[u] = N[v]$ by the definition of $B[v]$. Therefore, $B[u]$ contains each vertex from $B[v]$. In view of Lemma 1, $|B[v]| = \beta = |B[u]|$, and we have the equality $B[u] = B[v]$. Hence, $B[u] \subseteq N[u]$ and u is of type (B) for any $u \in B[v]$.

(2) Let v be a vertex of type (C) in Γ . Let $u \in B(v)$ and $B[u] \neq B[v]$. By the definition of a vertex of type (C), there exists a unique vertex $v^* \in N(v) \cap B(v)$ such that $N[v] = N[v^*]$. Thus, $u \neq v^*$ and $u \notin N[v]$. Furthermore, $N(u) \cap N(v) = N(v) \setminus \{v^*\}$.

By Lemma 7 and by the previous statement, a vertex u can not be of types (A) and (B). Hence, u is of type (C) and there exists $u^* \in N(u) \cap B(u)$ such that $N[u] = N[u^*]$.

Let $w \in B[u] \setminus B[v]$. Clearly, $u^* \in B(v)$ and, therefore, $w \neq u^*$. Then $N(w) \cap N(u) = N(u) \setminus \{u^*\} = N(u) \cap N(v) = N(v) \setminus \{v^*\}$. But this contradicts $w \notin B[v]$. Hence, $B[u] = B[v]$ and u is a vertex of type (C). \square

Lemma 13. *For any vertices v and u of Γ , if $B[v] \cap B[u] \neq \emptyset$, then $B[v] = B[u]$.*

Proof. This lemma is a direct corollary of Lemmas 7 and 12. \square

Lemma 14. *If v is a vertex of type (B) or (C) of Γ , then for any vertex $u \in N(v) \setminus B(v)$ we have $B[u] \subseteq N(v) \setminus B(v)$.*

Proof. If v is a vertex of type (B) or (C) in Γ , then we have $N[v] = N[v']$ for any $v' \in N[v] \cap B[v]$. Let $u \in N(v) \setminus B(v)$. In this case, $N(u)$ contains $N[v] \cap B[v]$. Let $u' \in B(u)$ and $u' \notin N(v) \setminus B(v)$. By Lemma 13, $u' \notin B[v]$. Therefore, $u' \notin N[v]$. But u' is adjacent to $k-1$ vertices from $N(u)$. Since $|N[v] \cap B[v]| > 1$, we find that $N(u') \cap B(v) \neq \emptyset$ and u' is adjacent to some vertex $v' \in B[v]$. This fact contradicts $N[v] = N[v']$ for any $v' \in N[v] \cap B[v]$. \square

Lemma 15. *Let v be a vertex of Γ . Then the following statements hold.*

- (1) *If v is of type (A1) or (C) in Γ , then $\beta + 1$ divides $k - 1$.*
- (2) *If v is of type (A2) or (B) in Γ , then $\beta + 1$ divides $k - \beta$.*

Proof. By Lemma 13, the sets $B[v]$ and $B[u]$ either coincide or do not intersect for any two vertices u and v . Since $|B[v]| = |B[u]| = \beta + 1$, statements (1) and (2) follow from Lemma 11 and Lemma 14. \square

Lemma 16. *Types of all the vertices of Γ satisfy only statement (1) or only statement (2) of Lemma 15.*

Proof. If Γ contains vertices of types satisfying different statements of Lemma 15, then $\beta + 1$ divides both $k - 1$ and $k - \beta$. But this contradicts $\beta > 1$. \square

Lemma 17. *A graph Γ all of whose vertices are of type (B) does not exist.*

Proof. Let all the vertices of Γ be of type (B) and $v \in V(\Gamma)$. By Lemma 14, if $u \in N(v) \setminus B(v)$, then the set $B[u]$ is contained in $N(v) \setminus B(v)$.

For any u in $(N(v_1) \setminus B(v_1)) \cap (N(v_2) \setminus B(v_2))$ the set $B[u]$ is contained in $(N(v_1) \setminus B(v_1)) \cap (N(v_2) \setminus B(v_2))$.

If v_1 is adjacent to v_2 , then $N(v_1) \cap N(v_2)$ contains $B(v_1)$, $B(v_2)$, and $B[u]$ for any $u \in (N(v_1) \setminus B(v_1)) \cap (N(v_2) \setminus B(v_2))$. Hence, by Lemma 13, $\beta + 1$ divides $|N(v_1) \setminus B(v_1)) \cap (N(v_2) \setminus B(v_2))| = a - 2\beta$.

If v_1 is not adjacent to v_2 , then $(N(v_1) \cap N(v_2)) \cap (B(v_1) \cup B(v_2)) = \emptyset$. Hence, by Lemma 13, $\beta + 1$ divides $|N(v_1) \cap N(v_2)| = a$.

Since Γ is a strictly Deza graph, we have both possibilities for v_1 and v_2 to be adjacent or to be nonadjacent. Comparing these two cases, we have $\beta + 1$ divides 2β . However, this contradicts the condition $\beta > 1$. \square

Lemma 18. *There are no vertices of type (A1) in Γ .*

Proof. Let a vertex x be of type (A1) in Γ . Fix the vertex x . In view of Lemma 16, each vertex u in Γ is either of type (A1) or of type (C).

By Lemma 9, not only for x but for each vertex u of Γ of type (A1), we have a unique vertex $u^* \in N(u)$ such that $N(u^*) \cap B(u) = \emptyset$. Moreover, by Lemma 11(1), for any vertex $w \in N(u) \setminus \{u^*\}$ we have $B[w]$ is contained in $N(u) \setminus \{u^*\}$.

By the definition, if a vertex u of Γ is of type (C), then $|N(u) \cap B(u)| = 1$. We denote the vertex from $N(u) \cap B(u)$ by u_* . In this case, we have $N[u] = N[u_*]$. Moreover, by Lemma 14, for any vertex $w \in N(u) \setminus \{u_*\}$ we have $B[w]$ is contained in $N(u) \setminus \{u_*\}$.

By Lemma 4(1), $N(x) \cap N(x^*) \neq \emptyset$. Let $u \in N(x) \cap N(x^*)$. Seeing the vertex x is of type (A1), we have $u \notin B[x]$ and $|N(x) \cap N(u)| = a$.

If u is of type (C), then for each vertex $w \in (N(x) \cap N(u)) \setminus \{x^*, u_*\}$ the set $B[w]$ is contained in $(N(x) \cap N(u)) \setminus \{x^*, u_*\}$. Hence, $|B[w]| = \beta + 1$ divides $|N(x) \cap N(u) \setminus \{x^*, u_*\}|$ which is equal to $a - 2$.

If u is of type (A1), then for each vertex $w \in (N(x) \cap N(u)) \setminus \{x^*, u^*\}$ the set $B[w]$ is contained in $(N(x) \cap N(u)) \setminus \{x^*, u^*\}$. Hence, $|B[w]| = \beta + 1$ divides $|N(x) \cap N(u) \setminus \{x^*, u^*\}|$ which is equal to $a - 2$ if u^* is adjacent x or is equal to $a - 1$ if u^* is nonadjacent x .

By lemma 8(2), $B[w]$ is contained in $N(x) \cap N(x^*)$ for any vertex w from $N(x) \cap N(x^*)$. Hence, by lemma 13, $|B[w]| = \beta + 1$ divides $|N(x) \cap N(x^*)| = a$. On the other hand, $\beta + 1$ divides $a - 2$ or $a - 1$. However, this contradicts $\beta > 1$. \square

Lemma 19. *There are no vertices of type (A2) in Γ .*

Proof. Let x be a vertex of type (A2) in Γ . Fix the vertex x . In this case, by Lemma 16, any vertex v of Γ is of type (A2) or (B).

By Lemmas 9 and 11(2), not only for x but for each vertex u of Γ of type (A2), we have only vertex $u^* \notin N(u)$ such that $B(u^*) \subseteq N(u)$. Moreover, by Lemma 11(2) for any vertex $w \in N(u) \setminus B(u^*)$ we have $N(u) \setminus B(u^*)$ contains $B[w]$.

By the definition, if a vertex u of Γ is of type (B), then $B(u) \subseteq N(u)$. Moreover, by Lemma 14 for any vertex $w \in N(u) \setminus B(u)$ we have $N(u) \setminus B(u)$ contains $B[w]$.

By Lemma 4(1), $N(x) \cap N(x^*) \neq \emptyset$. Let $u \in N(x) \cap N(x^*)$. Seeing the vertex x is of type (A2), in view of Lemma 8, we have $u \notin B[x]$, $|N(x) \cap N(u)| = a$, and $B(x^*) \subseteq N(x) \cap N(u)$.

If u is of type (B), then $(B(x^*) \cup B(u)) \subseteq (N(x) \cap N(u))$. Moreover, for each vertex $w \in (N(x) \cap N(u)) \setminus (B(x^*) \cup B(u))$ the set $B[w]$ is contained in $(N(x) \cap N(u)) \setminus (B(x^*) \cup B(u))$. Hence, $|B[w]| = \beta + 1$ divides $|N(x) \cap N(u) \setminus (B(x^*) \cup B(u))|$ which is equal to $a - 2\beta$.

If u is of type (A2), then $B(x^*) \subseteq N(x) \cap N(u)$. By Lemma 11(2), $B(u^*) \subseteq N(x) \cap N(u)$ or $B(u^*) \cap N(x) \cap N(u) = \emptyset$. Moreover, for each vertex $w \in (N(x) \cap N(u)) \setminus (B(x^*) \cup B(u^*))$ the set $B[w]$ is contained in $(N(x) \cap N(u)) \setminus (B(x^*) \cup B(u^*))$. Hence, by Lemma 13, $|B[w]| = \beta + 1$ divides $|N(x) \cap N(u) \setminus (B(x^*) \cup B(u^*))|$, which is equal to $a - 2\beta$ or $a - \beta$.

By lemmas 8(2) and 13, $\beta + 1$ divides $|N(x) \cap N(x^*)| = a$. On the other hand, $\beta + 1$ divides $a - 2\beta$ or $a - \beta$. However, this contradicts $\beta > 1$. \square

Lemma 20. *If all vertices in Γ are of type (C), then Γ is isomorphic to the 2-clique extension of a complete multipartite graph with parts of size $\frac{n-k+1}{2}$.*

Proof. Let each vertex in Γ be of type (C). Consider the relation ρ on the set of all vertices of Γ setting $v \rho u$ if and only if $u \in B[v]$ in Γ .

By Lemma 13, the relation ρ is an equivalence relation.

Let us consider a quotient graph Γ/ρ setting $B[v]$ adjacent to $B[u]$ if and only if v adjacent to u . By Lemma 14, the map $v \rightarrow B[v]$ from Γ on Γ/ρ preserves the adjacency of vertices.

The quotient graph $\Gamma_\rho = \Gamma/\rho$ has the vertex set $\{B[v] | v \in \Gamma\}$. Thus, Γ_ρ has exactly $\frac{n}{\beta+1}$ vertices. Moreover, in view of Lemma 14, the degree of each vertex is equal to $\frac{k-1}{\beta+1}$.

Let $v_\rho = B[v]$ and $u_\rho = B[u]$ be distinct vertices of Γ_ρ . According to the equivalence relation ρ , there are only two possibilities for the number of vertices in $N(v_\rho) \cap N(u_\rho)$ in Γ_ρ . We use $\{v^*, u^*\}$ in the same sense as well as in the proof of Lemma 18.

(1) Let a vertex u do not belong to $N[v] \cup B[v]$. In this case, $N(v) \cap N(u)$ contains $B[w]$ for any $w \in (N(v) \cap N(u)) \setminus \{v^*, u^*\}$. But $N(v) \cap N(u)$ does not contain v^* and u^* . Hence, the vertices v_ρ and u_ρ have exactly $\frac{a}{\beta+1}$ common adjacent vertices in Γ_ρ .

(2) Let a vertex u belong to $N[v] \setminus B[v]$. In this case, $N(v) \cap N(u)$ contains both v^* and u^* , and $B[w]$ for any $w \in (N(v) \cap N(u)) \setminus \{v^*, u^*\}$. Hence, the vertices v_ρ and u_ρ have exactly $\frac{a-2}{\beta+1}$ common adjacent vertices in Γ_ρ .

Since $\beta > 1$, these two cases can not be implemented together.

If case (1) occurs for the vertices of Γ , then it follows that $N[v] \setminus B[v] = \emptyset$. However, $N[v] \setminus B[v] = N(v) \setminus \{v^*\}$, and so $k = 1$, a contradiction, because Γ is a strictly Deza graph and has diameter 2.

Suppose that case (2) occurs for the vertices of Γ . Then the quotient graph Γ_ρ is a complete graph on $\frac{n}{\beta+1}$ vertices and $B[v]$ is the union of $\frac{(\beta+1)}{2} = \frac{n-k+1}{2}$ disjoint edges.

Hence that graph Γ is a extension of the complete graph on $\frac{n}{\beta+1}$ vertices with t disjoint copies of K_2 , there $t = \frac{(\beta+1)}{2}$. Thus, Γ is isomorphic to the 2-clique extension of a complete multipartite graph with parts of size $t = \frac{(\beta+1)}{2} = \frac{(n-k+1)}{2}$. \square

We proved that if a strictly Deza graph Γ satisfies the condition of Theorem 1, then Γ cannot contain only vertices of type (B) by Lemma 17. There are no vertices of type (A1) in Γ by Lemma 18 and there are no vertices of type (A2) in Γ by Lemma 19.

Hence, all the vertices in Γ are of type (C). By Lemma 20, Γ satisfies the conclusion of Theorem 1.

Theorem 1 is proved.

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